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Unified Theory of Linear Noisy Two-Ports

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***Abstract*—Network noise invariants are introduced that lead to improved noise characterization and a complete theory of linear noisy two-ports. Minimum power added noise temperature and minimum cold load temperature are identified as network noise invariants under lossless embedding. Associated invariant equations provide explicit relations between all known and new network invariants. From these equations, an invariant under lossless embedding is identified that defines network noise–gain coupling in the most basic terms of noise correlation, minimum noise temperature, and complex nonreciprocal gain. A noise correlation parameter *q* is formally introduced that is invariant to lossless input and/or output transformation. Conditions and bounds are established, and it is shown that *q* ≈ 2 for low-noise active devices. An exact expression for the *q* parameter of a minimum noise cascade network is given in terms of constituent device invariants. From a systems point of view, the cascade *q* parameter represents source impedance noise sensitivity. A lower bound on cascade *q* is determined by device invariants minimum power added noise temperature and minimum cold load temperature. It is shown that the cascade *q* lower bound is realized by simultaneous noise and power match.**

***Index Terms*—Active cold load, low-noise amplifier (LNA), noise measure, noise parameter, noise temperature, noise theory.**

I. Introduction

An INVARIANT quantity identifies a fundamental property of a physical system that is unaltered by mere changes to that system's environment. Invariants often serve as a figure of merit or benchmark value for comparing physical systems, verifying computations or measurements, or establishing theoretical performance bounds. They can also provide insight and better understanding of the operating principles and behavior for the system to which they apply. The purpose of this paper is to identify and discuss several new invariants of linear noisy two-port networks, leading to a unified noise theory.

The first two-port invariant came in 1954 when Mason [1] deduced an invariant network property under lossless embedding and identified it as *maximum unilateral power gain* U. A discussion of unilateral gain and invariance can be found in a 1992 tutorial review by Gupta [2]. In 1962, Rollett [3] introduced an invariant *stability factor* k and discussed important invariant power gains, *maximum available gain* G_{ma} and *maximum stable gain* G_{ms}.

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In order to describe the behavior of a linear two-port containing internal noise sources, Rothe and Dahlke [4] introduced the "Theory of Noisy Fourpoles" in 1956 which showed that four noise parameters are required. One such set of noise parameters consists of: 1) minimum noise temperature T_{emin}; 2) optimum source conductance G_{opt}; 3) optimum source susceptance B_{opt}; and 4) noise resistance R_n. In 1958, following shortly after the works of Mason and Rothe and Dahlke, Haus and Adler [5] published the "Optimum Noise Performance of Linear Amplifiers," which identified a two-port noise invariant under lossless embedding called *minimum noise measure* M_{min}. Unlike minimum noise temperature T_{emin}, this amplifier figure of merit accounts for gain and noise in a way that does not change with feedback. M_{min} defines the best possible noise performance for a high-gain cascade of identical amplifiers.

Since the discovery of signal invariant U and noise invariant M_{min}, there have been two fundamental additions to two-port noise characterization. First, it was shown by Lange [6] in 1967 that the product R_nG_{opt} is invariant under the more restrictive condition of lossless input transformation. He proposed the parameter $N = R_nG_{opt}$ to replace R_n, recognizing that the invariant property of N makes it a more fundamental quantity than R_n. Secondly, Wiatr [7], [8] first observed in 1980 that the relation $4NT_0/T_{emin} \geq 1$ must be satisfied to represent noise of a physical two-port. (T_0 is the reference temperature 290 K.) Thus, a theoretical bound was established on the relative value of two noise parameters, the ratio of N and T_{emin}.

In this paper, we introduce new network invariants, physical interpretations, and invariant relations that form a complete and unified two-port noise theory. Section II of the paper discusses the two independent noise invariants under lossless embedding that relate all other noise, gain, and stability invariants of the two-port. Physical interpretations of these two invariants provide a deeper understanding of network operation than do noise temperature or noise measure. In Section III, we reveal the significance and true nature of the quantity $4NT_0/T_{emin}$ by formal introduction as *noise correlation parameter* q. We will prove the two sufficient conditions taken together for $q \leq 2$, irrespective of network topology or number of noise sources, and furthermore show that $q \approx 2$ *for any low-noise amplifier device*. Section IV applies new knowledge of two-port invariants to the high-gain cascade low-noise amplifier case. We derive the lower bound of cascade q parameter, expressed in terms of constituent amplifier general invariants, then show that the lower bound is realized by simultaneous noise and power match of the individual

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amplifiers. Section V concludes the paper with a summary and discussion of noise theory results.

II. Network Noise Invariants

There are two independent noise power invariants (expressed as temperature) associated with a two-port network. The first of these that we will examine is simply related to minimum noise measure. However, the physical interpretation of this invariant is much more revealing because it describes performance of an *individual amplifier* rather than a high-gain cascade. In addition, it explains clearly why noise temperature fails to describe fundamental amplifier noise performance.

A. Power Added Noise Temperature

The invariant we introduce is *minimum power added noise temperature* $(MT_0)_{\min}$. To understand the physical significance of this interpretation, consider the maser amplifier power flow diagram of Fig. 1. Available input signal power to the maser is S_i. Maser output consists of amplified signal G_aS_i plus added maser noise N_o. With a depleted lower energy state, no signal power is absorbed and there is total incident signal reflection at the maser port. The output signal is partitioned into two parts: 1) reflected input signal power S_i; and 2) signal power added by the stimulated emission process, $(G_a - 1)S_i$. Only *added* signal and noise powers generated by internal maser processes determine its true noise performance. Indeed, for any amplifier, available input signal power must be excluded from consideration. *Inclusion of available input signal power is why traditional noise temperature fails to define fundamental amplifier noise performance*.

Power added noise temperature MT_0 is easily obtained by referring output noise to the input according to $N_o/(G_a - 1)$ and applying the definition of noise measure. We retain the noise measure M notation to show explicitly its relation to power added noise temperature and to acknowledge the work of Haus and Adler. Lastly, it is easy to show that power added noise temperature is preserved for any cascade, independent of gain or gain order. Thus, optimum high-gain cascade noise performance is simply a consequence of the individual amplifier invariant.

B. Cold Load Temperature

We identify *minimum cold load temperature* $T_{c\min}$ as the second two-port invariant under lossless embedding. It is the best possible noise performance of a two-port network operated as an *active cold load*. Fig. 2 shows an active two-port embedded in a passive lossless four-port network, which is then connected to a passive lossless three-port network. Invariance means that for all possible four-port embeddings, a three-port network can be found that gives minimum available output noise power $kT_{c\min}B$, where k is Boltzmann's constant and B is noise bandwidth. Note that we are interested only in active two-ports here since the output noise of any passive one-port network with loss is defined by its physical temperature.

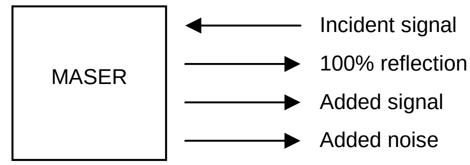

Fig. 1. Maser amplifier power flow diagram. This type of amplifier readily shows why *added* signal and noise power components define true noise performance—they alone result from internal device processes.

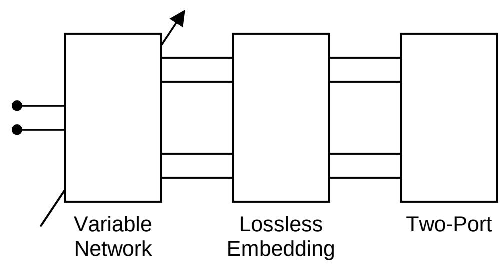

Fig. 2. Active cold load diagram illustrating the invariance of minimum cold load temperature under lossless embedding.

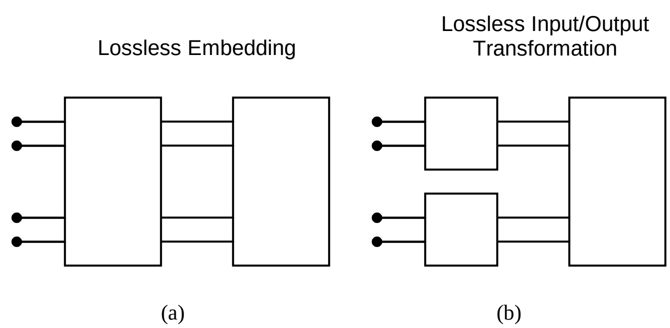

Fig. 3. Invariant conditions. (a) The *general* condition of lossless embedding includes feedback and defines fundamental invariance. (b) The *special* condition of lossless input/output transformation is simple impedance transformation at each port.

In view of Fig. 2, the invariance of minimum cold load temperature may seem obvious. However, the result is less obvious when the active network is generalized to an arbitrary number of active two-port devices. In a later publication, Haus and Adler [9] proved that eigenvalues are preserved, and thus also $T_{c\min}$, for n arbitrary active devices in a $4n$-port lossless embedding. This aspect of their work has received little attention, probably because the focus has been on amplifier noise performance. The proof will be important when we later derive a noise parameter lower bound for optimum cascade networks. A practical consequence is that active cold load performance cannot be improved by employing more low-noise devices.

C. Invariant Conditions

It is important to understand and distinguish between two different conditions of two-port invariants. In Fig. 3, *lossless* *embedding* comprises a passive lossless four-port connection to the two-port, whereas *lossless input/output transformation* connects separate uncoupled passive lossless two-port networks to the two-port. Lossless embedding includes feedback and is the more general condition. Therefore, an invariant subject to lossless embedding constitutes a more fundamental physical property of the two-port.

The tables below summarize noise, gain, and stability invariants of linear noisy two-ports discussed in this paper. Known invariants are shown in Table I and new invariants introduced are given in Table II. Expressions are in admittance parameter form. The invariant type is termed *general* for those subject to lossless embedding and *special* in the case of lossless input/output transformation. The condition will now be clear when the unmodified term "invariant" is used in the paper with these various quantities.

D. Invariant Equations

A modified form of the *characteristic noise matrix* given by Haus and Adler [5, eq. (41)] is

$$\mathbf{N} = \tfrac{1}{2}[\mathbf{P} - \mathbf{TPT}^{+}]^{-1}\mathbf{C} \tag{1}$$

with *permutation matrix* **P**, *general-circuit matrix* **T**, and *noise correlation matrix* **C** defined by

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

$$\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{3}$$

$$\mathbf{C} = T_{emin}\begin{bmatrix} \dfrac{q}{G_{\text{opt}}} & 2 - \dfrac{qY^{*}_{\text{opt}}}{G_{\text{opt}}} \\ 2 - \dfrac{qY_{\text{opt}}}{G_{\text{opt}}} & \dfrac{qY_{\text{opt}}Y^{*}_{\text{opt}}}{G_{\text{opt}}} \end{bmatrix}. \tag{4}$$

The symbol "*" means complex conjugate of a variable and the superscript "+" denotes conjugate transpose of a matrix. With **C** defined this way, the eigenvalues of characteristic noise matrix **N** are the invariants $(MT_0)_{\min}$ and $-T_{c\min}$. The relation of noise correlation matrix **C** to noise matrices of Haus and Adler [5, eq. (24)] and Hillbrand and Russer [10, eq. (11)] is, respectively,

$$kB\mathbf{C} = \overline{\boldsymbol{\delta\delta}^{+}} = 2B\boldsymbol{C}_{A}\,. \tag{5}$$

(There are two errors in the Hillbrand and Russer paper's correlation matrix equation (11): 1) Y_{opt} and Y^*_{opt} should be reversed [11]; and 2) absolute temperature T should be reference temperature T_0.)

Invariant equations are obtained by equating the product and sum of eigenvalues to determinant **N** and trace **N**, respectively. Initially, the equations are in terms of signal and noise parameters contained in (3) and (4). From these expressions, all invariant quantities are identified that relate to the general invariants $(MT_0)_{\min}$ and $T_{c\min}$. The *invariant equations* are shown here in final form.

$$(MT_0)_{\min} T_{c\min} = \frac{(q-1)T_{emin}^2}{1 - \dfrac{1}{G_{mi}}} \tag{6}$$

$$(MT_0)_{\min} - T_{c\min} = \frac{T_{emin}\Delta}{1 - \dfrac{1}{G_{mi}}} \tag{7}$$

where

$$\Delta = 2 - q + \frac{q}{G_{a\text{opt}}} - \frac{2k}{G_{ms}}\,. \tag{8}$$

The introduced *maximum invariant power gain* G_{mi} is similar to well-known maximum available gain G_{ma}, and nearly identical to Kotzebue's maximally efficient power gain G_{me} [12]. G_{mi} is in fact closely related to the U-function, and to the defining condition of two-port activity and passivity [13, pp. 135–141]. $G_{a\text{opt}}$ is available gain at minimum noise temperature, known as *associated gain*. Analytic expressions for all invariants in the above equations are given in Tables I and II.

E. Noise–Gain Coupling

As suggested above, the isolated gain G_{mi} appearing in the invariant equations does indeed have a special significance. Consider that G_{mi} is the only signal quantity present in (6). The right-hand side of (6) is itself an invariant quantity under lossless embedding. We thus have an expression of fundamental two-port noise–gain interaction with varying embedding circuit conditions. The denominator may be written as the product of a general invariant and a special invariant,

$$1 - \frac{1}{G_{mi}} = \left(1 - \frac{1}{U}\right)\cdot\left|1 - \frac{y_{12}}{y_{21}}\right|^2. \tag{9}$$

Substituting (9) into (6) and collecting general invariants on the left-hand side, we have

$$\left(1 - \frac{1}{U}\right)(MT_0)_{\min} T_{c\min} = \frac{(q-1)T_{emin}^2}{\left|1 - \dfrac{y_{12}}{y_{21}}\right|^2}\,. \tag{10}$$

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This is two-port *noise–gain coupling* in its most basic form. The right-hand side of (10) contains noise parameters q and T_{emin}, and the inverse of signal quantity y_{21}/y_{12} known as *complex nonreciprocal gain* [13, pp. 175–176].

For a high-gain active two-port device, $(MT_0)_{\min} \approx T_{\text{emin}}$ and $G_{mi} >> 1$ so that (6) can be approximated as

$$T_{\text{cmin}} \approx (q-1)T_{\text{emin}} \,. \tag{11}$$

When the device is also low-noise, we expect that q cannot approach unity since that would imply very low active cold load temperature—intuitively, an unlikely physical possibility. In the following section, we find this is indeed correct as an investigation of the q-parameter is undertaken.

III. Noise Correlation Parameter

It is evident from invariant equation (10) that the q-parameter plays a fundamental role in two-port network characterization. It is the remaining parameter of the network invariant equations to be understood in terms of physical meaning. We begin with the formal introduction of *noise correlation parameter q* defined by known noise parameters as

$$q = \frac{4R_n G_{\text{opt}} T_0}{T_{\text{emin}}} \,. \tag{12}$$

According to (12), q is invariant since $N = R_n G_{\text{opt}}$ and T_{emin} are invariant. From (6), $q < 1$ is not allowed to give negative invariant temperatures of an active network. Furthermore, limitations on the cold load temperature (11) indicate that q cannot approach unity for a high-gain low-noise device.

In this section, we will continue to develop a complete physical understanding of the q-parameter. We then address a broad class of two-port networks, the important case of low-noise active devices. The properties of a low-noise active network will be identified that lead to a constraint on the q-parameter value. In fact, we will show that $q \approx 2$ *for any low-noise amplifier device.*

A. Noise Characterization

Following Engberg and Larsen [14], the noise properties of a linear noisy two-port can be characterized by a *noise network* as shown in Fig. 4. The noise network consists of a noise voltage source e_n and a noise current source partitioned into correlated and uncorrelated parts $i_n = Y_\gamma e_n + i_u$, respectively. The *correlation admittance* Y_γ and mean-squared source values are defined by the following, which include *noise resistance* R_n and *noise conductance* G_n.

$$Y_\gamma = G_\gamma + jB_\gamma \tag{13}$$

$$\bar{e}_n^{\,2} = 4kT_0 R_n B \tag{14}$$

$$\bar{i}_u^{\,2} = 4kT_0 G_n B \tag{15}$$

Noise parameters are related to the above source parameters by

$$T_{\text{emin}} = 2R_n T_0 (G_\gamma + G_{\text{opt}}) \tag{16}$$

$$G_{\text{opt}} = \left(\frac{G_n}{R_n} + G_\gamma^2 \right)^{1/2} \tag{17}$$

$$B_{\text{opt}} = -B_\gamma \,. \tag{18}$$

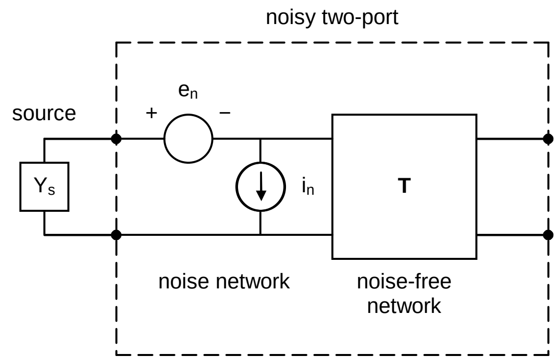

Fig. 4. Chain representation of linear noisy two-port with partially correlated sources e_n and i_n and general-circuit matrix **T**.

Substituting (16) into the defining equation (12) we have

$$q = \frac{2}{1 + \dfrac{G_\gamma}{G_{\text{opt}}}} \,. \tag{19}$$

Equation (19) expresses the correlation nature of q. The quantity G_γ/G_{opt} is the correlation coefficient between noise sources e_n and i_n when $B_\gamma = 0$ (otherwise the correlation is complex-valued). Since $G_n \geq 0$ and $R_n \geq 0$, we can see from (17) that the range of this real-valued and invariant correlation quantity is

$$-1 \leq \frac{G_\gamma}{G_{\text{opt}}} \leq 1 \,. \tag{20}$$

The resulting range of noise correlation parameter q is described in Fig. 5. Note the three important cases of q value. When $G_\gamma = 0$, the noise sources e_n and i_n are *uncorrelated* and $q = 2$. When $G_n = 0$, we have either *full positive correlation* with $q = 1$ or *full negative correlation* with $q = \infty$. Fully correlated noise sources e_n and i_n represent a single physical noise source within a two-port network. Practical amplifier devices contain physical sources having both positive and negative correlation.

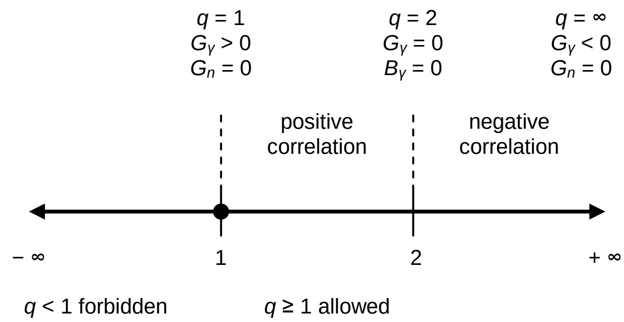

Fig. 5. Range of q-parameter values with correlation of noise network sources e_n and i_n.

Referring again to Fig. 4, the *effective input noise temperature* T_e of an active two-port is a function of *source admittance* $Y_s = G_s + jB_s$ according to the well-known relation

$$T_e = T_{e\min} + \frac{R_n T_0}{G_s}\left|Y_s - Y_{\mathrm{opt}}\right|^2 \tag{21}$$

with *optimum source admittance* $Y_{\mathrm{opt}} = G_{\mathrm{opt}} + jB_{\mathrm{opt}}$. Here we digress briefly to comment regarding the interpretation of this equation. It is common to find in the literature reference to R_n as a "sensitivity factor" based on its appearance in (21). This has led, for example, to semiconductor device design for low R_n in order to reduce receiver noise temperature variation due to antenna impedance change. Neither of these ideas is correct, however. The problem is that the form of (21) does not explicitly separate device noise characteristics from impedance mismatch effects. The R_n parameter is not invariant, but subject to input transformation and superficial semiconductor device treatment, neither of which affect two-port impedance sensitivity. Using (12) to substitute for R_n in (21), we obtain the following form with device noise and mismatch effects separated, as evidenced by the symmetry in admittance quantities.

$$T_e = T_{e\min} + qT_{e\min}\frac{\left|Y_s - Y_{\mathrm{opt}}\right|^2}{4G_s G_{\mathrm{opt}}} \tag{22}$$

Clearly, it is noise correlation parameter q that defines sensitivity to impedance mismatch. Our anticipated result $q \approx 2$ for low-noise devices means that this value must be used for receiving system optimization and cannot be changed by semiconductor device design.

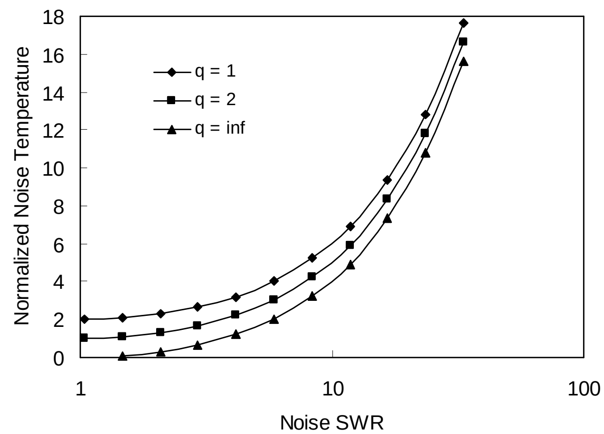

Fig. 6. Noise temperature variation with source impedance showing effect of network correlation. The rms values of noise voltage e_n and noise current i_n are held constant for the three limiting cases of network q value.

We now express (22) in a more convenient form and examine the effect of correlation on noise temperature in the three limiting cases q = 1, 2, and ∞. The *noise equation* in terms of standing-wave ratio (SWR) is

$$T_e = T_{e\min} + qT_{e\min}\frac{(S-1)^2}{4S} \tag{23}$$

where S is the *noise SWR* of source admittance Y_s with respect to optimum source admittance Y_{opt}. If we normalize $T_{e\min}$ to that of the uncorrelated q = 2 case, then $qT_{e\min} = 4R_n G_{\mathrm{opt}} T_0 = 2$ for all cases. This is correct because we hold the rms values of noise voltage e_n and noise current i_n constant, and therefore R_n and G_{opt} are constant according to (14), (15), and (17). $T_{e\min}$ is determined by available noise power from the noise network at the noise-free network side in Fig. 4.

Fig. 6 shows noise temperature comparison for the three limiting cases of full positive correlation (q = 1), uncorrelated (q = 2), and full negative correlation (q = ∞). The most important feature of these results is that the noise temperature difference between adjacent plotted cases is a constant value of uncorrelated $T_{e\min}$ over all SWR. Consequently, noise temperature T_e is nearly independent of q value at high SWR, where large available noise power of one source, either e_n or i_n, dominates. This fact will be important when we analyze two-ports having multiple physical noise sources.

The influence of q is greatest for the *noise matched* condition $Y_s = Y_{\mathrm{opt}}$ where S = 1. The relative values of $T_{e\min}$ at S = 1 suggest a physical definition of q in terms of available noise power. Applying the factor kB to (12) and substituting (14) gives

$$q = \frac{\overline{e}_n^2 G_{\mathrm{opt}}}{kT_{e\min} B}\,. \tag{24}$$

Thus, q is the ratio of minimum available power from the noise network at full positive correlation to actual minimum available power.

The limiting case of full negative correlation ($q = \infty$) deserves comment here since it has $T_{emin} = 0$. A low-noise amplifier operating at this condition or near to it would be very sensitive to source impedance, with noise temperature reaching many times the matched value when $S = 2$, for example. (In Fig. 6, noise is 25 percent of the uncorrelated T_{emin} when $S = 2$.) The situation is identical to that of an active cold load with $q = 1$ and a similar comment (Section II-E) applies: noise levels observed in unmatched amplifiers would imply a very low and physically unreasonable T_{emin} (e.g., below quantum limit) if the q value is large.

So if the q value of a low-noise amplifier cannot approach the limits of $q = 1$ or $q = \infty$, then what must the value of q be? With the foregoing noise characterization background in place, we are ready to answer that question and begin the details of proof for $q \approx 2$.

B. *Network Conditions for* $q \leq 2$

We now prove the general network conditions for $q \leq 2$ that serve as a useful reference point for the low-noise amplifier case. The two conditions for $q \leq 2$, irrespective of network topology or number of noise sources, are as follows:

1) *All noise sources within the network are mutually uncorrelated.*
2) *Each noise source within the network can be represented by a noise network having* $q \leq 2$.

"Noise source" here can be a single physical source within the network or a subset of physical sources represented by a noise network. As we shall see, these conditions are *sufficient, but not necessary* for $q \leq 2$.

Consider two uncorrelated noise sources represented by noise networks having parameters q_1, T_{emin1}, Y_{opt1} and q_2, T_{emin2}, Y_{opt2}. We wish to find the noise parameters for these combined noise networks, in particular the combined noise correlation parameter q for the two noise sources. The problem can be simplified by applying simultaneous lossless input transformation, with no loss of generality since q and T_{emin} are invariant. In the reflection coefficient plane (Smith chart), the result of this transformation is *optimum source impedances* R_{opt1} and R_{opt2} with combined R_{opt} in between, all located on the real line.

To determine q, we evaluate noise equations in the form of (23) for the two individual sources and their combination. The equations may be evaluated at source impedances anywhere on the Smith chart complex reflection coefficient plane, but for simplicity we choose to evaluate at *source impedances* R_s on the real line with noise SWRs R_s/R_{opt1}, R_s/R_{opt2}, and R_s/R_{opt}, respectively. Using (23), we thus have

$$T_{e1} = T_{emin1} - \frac{q_1 T_{emin1}}{2} + \frac{q_1 T_{emin1} R_s}{4R_{opt1}} + \frac{q_1 T_{emin1} R_{opt1}}{4R_s} \tag{25}$$

$$T_{e2} = T_{emin2} - \frac{q_2 T_{emin2}}{2} + \frac{q_2 T_{emin2} R_s}{4R_{opt2}} + \frac{q_2 T_{emin2} R_{opt2}}{4R_s} \tag{26}$$

$$T_e = T_{emin} - \frac{q T_{emin}}{2} + \frac{q T_{emin} R_s}{4R_{opt}} + \frac{q T_{emin} R_{opt}}{4R_s}. \tag{27}$$

Since the two individual noise sources are uncorrelated, their noise temperatures add and we can also write

$$T_e = T_{e1} + T_{e2}. \tag{28}$$

Equations (25)–(28) are solved for combined q in terms of q_1, q_2, source *separation ratio* α, and source *strength ratio* β, where

$$\alpha = \frac{R_{opt2}}{R_{opt1}} \tag{29}$$

$$\beta = \frac{q_2 T_{emin2}}{q_1 T_{emin1}}. \tag{30}$$

The solution proceeds by substituting (25) and (26) into (28), differentiating with respect to R_s, and equating to zero to give

$$R_{opt} = \left[\frac{(1+\beta\alpha) R_{opt1} R_{opt2}}{\alpha + \beta}\right]^{1/2}. \tag{31}$$

Then, setting $R_s = R_{opt}$ in (25) and (26), we obtain via (28)

$$\frac{T_{emin}}{q_1 T_{emin1}} = \frac{1}{q_1} - \frac{1}{2} + \frac{R_{opt}}{4R_{opt1}} + \frac{R_{opt1}}{4R_{opt}} + \beta\left(\frac{1}{q_2} - \frac{1}{2} + \frac{R_{opt}}{4R_{opt2}} + \frac{R_{opt2}}{4R_{opt}}\right). \tag{32}$$

Lastly, derivatives of (27) and (28) with respect to R_s are equated, and at $R_s = R_{opt2}$ we obtain

$$\frac{1}{q} = \frac{T_{emin}\left(\frac{R_{opt2}}{R_{opt}} - \frac{R_{opt}}{R_{opt2}}\right)}{q_1 T_{emin1}\left(\alpha - \frac{1}{\alpha}\right)}. \tag{33}$$

Substituting (31) and (32) into (33) and reducing yields the following equation for the combined q parameter of two uncorrelated sources:

$$\frac{1}{q} = \left[\frac{1}{q_1} - \frac{1}{2} + \beta\left(\frac{1}{q_2} - \frac{1}{2}\right)\right] \cdot \left[\frac{\alpha}{(\alpha+\beta)(1+\beta\alpha)}\right]^{1/2} + \frac{1}{2}\,. \quad (34)$$

The condition we seek to prove is q_1, $q_2 \le 2$. Since $\alpha > 0$ and $\beta > 0$ in (34), at least one of q_1 or $q_2 > 2$ is required for the product term to be negative and give $q > 2$. This proves that q_1, $q_2 \le 2$ is a sufficient condition for combined $q \le 2$ of *two* uncorrelated sources. The proof is extended to an arbitrary number of sources by induction. A third uncorrelated source having noise network parameter $q_3 \le 2$ is added to the combined source having $q \le 2$. These meet the conditions proven for two sources and thus form another combined network with $q \le 2$. This completes the proof that a two-port network comprising an arbitrary number of mutually uncorrelated sources or sets of sources, each having $q \le 2$, has an upper bound of noise correlation parameter $q = 2$.

An overview of the q-parameter behavior when sources combine is most readily seen by plotting (34). Fig. 7 shows a family of curves with strength ratio β as the parameter. Inspection of (24) shows that β is the relative available power of the two sources. When the power level of one source dominates, we see that q approaches the value of that source as an upper or lower limit. Extreme values of q (maximum or minimum) occur at $\alpha = 1$. As the separation ratio α increases, the two sources *decorrelate* and q approaches 2. This decorrelation occurs because at large α, noise voltage e_n from one source dominates while uncorrelated noise current i_n from the other source is dominant. Large available power differences between sources require greater separation for decorrelation.

Fig. 8 plots the q parameter for equal strength sources, i.e., with $\beta = 1$, the condition of greatest source interaction. This family of curves has q_1-q_2 pair values as the parameter. When $q_1 = q_2$, the extreme value at $\alpha = 1$ is $q = q_1 = q_2$ (true also for any β). Several unequal q_1-q_2 values are plotted, and of particular interest are the 1-∞ and 2-∞ cases. The first case 1-∞ represents single physical sources having full positive and negative correlation, respectively. The combined q value of these is 2 for all α, and demonstrates that the condition q_1, $q_2 \le 2$ is not necessary for $q \le 2$. The second case 2-∞ combines a decorrelated group of physical sources with a single physical source having full negative correlation. Here, the combined q is maximum $q = 4$ at $\alpha = 1$, and quickly decorrelates toward $q = 2$ as separation increases.

Setting $q = 2$ and $\beta = 1$ in (34) and solving for q_2 say, we obtain the $q = 2$ boundary for equal strength uncorrelated sources

$$\frac{1}{q_2} = 1 - \frac{1}{q_1}\,. \quad (35)$$

The boundary q_1-q_2 pair values give $q = 2$ for all separation α. This curve is plotted in Fig. 9 and emphasizes once again that the restrictive condition q_1, $q_2 \le 2$ is not required for combined $q \le 2$.

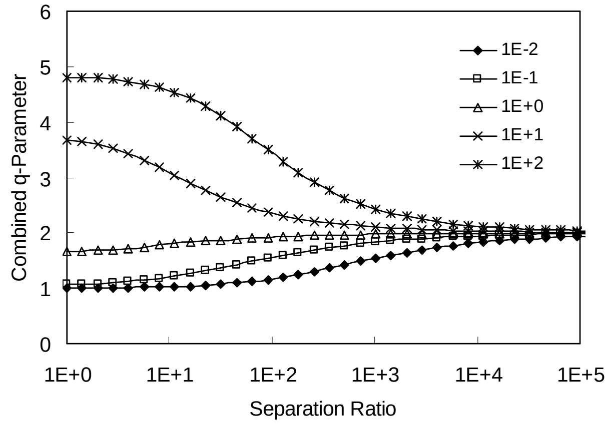

Fig. 7. Combined correlation of two uncorrelated sources with $q_1 = 1$, $q_2 = 5$, and parameter β, the source strength ratio.

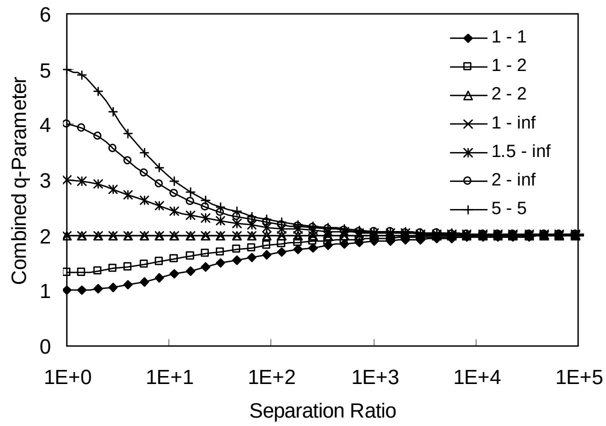

Fig. 8. Combined correlation of two uncorrelated sources with equal source strength $\beta = 1$ and parameter q_1 - q_2.

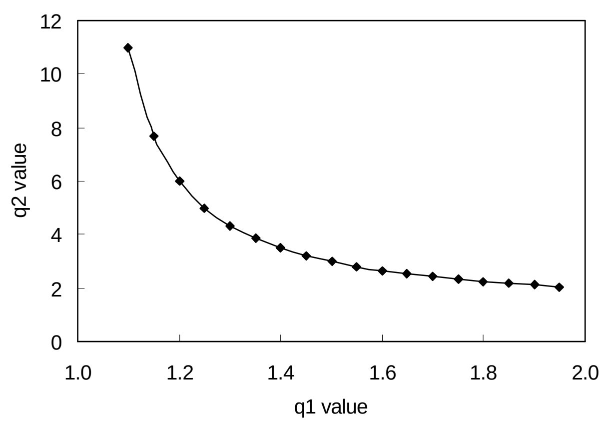

Fig. 9. Boundary of combined $q = 2$ for two equal-strength uncorrelated sources. Below the boundary curve, the q_1 and q_2 source values give combined $q < 2$.

The above examples indicate how physical sources in a noisy two-port network combine to give $q \leq 2$. Suppose a two-port contains a majority of physical sources with $q = 1$ and a small number of sources with negative correlation $q = \infty$. The general source condition $q \leq 2$ does not hold. However, if strengths of the negative sources do not exceed those of an equal number of positive sources, then we have 1-∞ pairs with $q \leq 2$ and it is known immediately that $q \leq 2$ for the entire two-port network. In the following section, we will use these physical source combining properties and an active two-port decorrelation property to prove $q \approx 2$ for any low-noise amplifier.

C. *Correlation Parameter of Low-Noise Amplifier*

An amplifier device contains a collection of *local physical noise sources* that are mutually uncorrelated. Thus, noise sources of the physical electrical network representation of an amplifier device are mutually uncorrelated. Each of these individual noise sources can be represented by a noise network having either $q = 1$ or $q = \infty$. We now identify the network conditions that constitute a *low-noise* amplifier and the accompanying network property that forces decorrelation of individual noise sources.

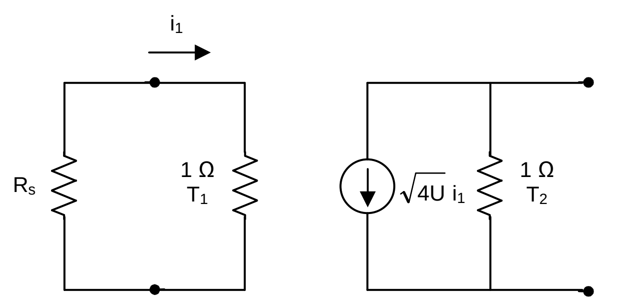

Fig. 10. Basic unilateral two-port amplifier with source resistance connected at the input.

Consider the basic unilateral two-port shown in Fig. 10 having input and output terminations $R_1 = R_2 = 1\ \Omega$, respectively, and controlled-source current gain $\sqrt{4U}$. The input termination R_1 is passive with thermal noise determined by its physical temperature T_1. The output termination R_2 of the active device under nonequilibrium condition has thermal noise determined by *equivalent* physical temperature $T_2 >> T_1$. The input circuit contribution of R_1 to amplifier noise temperature is found by equating input termination noise voltage to that of source resistance R_s at temperature T_{e1}, which gives

$$T_{e1} = \frac{T_1 R_1}{R_s}. \tag{36}$$

Output circuit noise temperature contribution of R_2 is

$$T_{e2} = \frac{T_2}{G_a}. \tag{37}$$

We see that noise temperature contribution T_{e1} is minimized by a large value of source resistance $R_s \rightarrow \infty$ that prevents noise current at the input. On the other hand, noise temperature contribution T_{e2} is minimized by power match at the input with $R_s = 1\ \Omega$. The large separation ratio of these optimum source resistance values results in decorrelation and combined $q = 2$. Thus, we state the following network principle:

Opposing input and output optimum source conditions of power mismatch and power match, respectively, is the fundamental reason for $q \approx 2$ in any low-noise amplifier.

The network conditions that define a low-noise amplifier and lead to this property are as follows. First, there must be substantially no dissipative losses preceding the input termination of the basic amplifier network. These losses, of course, increase noise temperature, but also force the optimum source toward the power match condition and $q = 1$. (The latter is easily seen by considering the limiting case of large attenuation, or by amplifier simulation with shunt resistor loss at the input.) The second condition necessary for low-noise amplifier is sufficient unilateral gain to overcome the output temperature T_2.

We can find the quantitative conditions that define a low-noise amplifier from the noise temperature of the basic amplifier network. The noise parameters of the output noise source are

$$q_2 = 1 \tag{38}$$

$$T_{e\min 2} = \frac{T_2}{U} \tag{39}$$

$$R_{\text{opt}2} = R_1. \tag{40}$$

The amplifier noise temperature is the sum of input noise contribution (36) and output noise contribution written in the form of (27) by using (38)–(40).

$$T_e = \frac{T_1 R_1}{R_s} + \frac{T_2}{2U} + \frac{T_2 R_s}{4UR_1} + \frac{T_2 R_1}{4UR_s} \tag{41}$$

Differentiating (41) with respect to R_s and equating to zero, we obtain the optimum source resistance relation

$$\frac{R_{\text{opt}}}{R_1} = \left(\frac{4UT_1}{T_2} + 1\right)^{1/2}. \tag{42}$$

According to (36), $R_{\text{opt}} >> R_1$ is required for a low-noise amplifier. From (42), we therefore have

$$4U >> \frac{T_2}{T_1}. \quad (43)$$

Since $T_2 >> T_1$ for an active device, a very large unilateral gain U is required for low-noise amplification.

To prove $q \approx 2$ for a low-noise amplifier, we find the mean-squared values of input and output noise network sources, respectively, given by

$$\overline{e}_{n1}^2 = 4kT_1R_1B \quad (44)$$

$$\overline{i}_{n1}^2 = 0 \quad (45)$$

$$\overline{e}_{n2}^2 = \frac{4kT_2R_1B}{4U} \quad (46)$$

$$\overline{i}_{n2}^2 = \frac{4kT_2B}{4UR_1}. \quad (47)$$

Rearranging (43), we have the inequality $T_1 >> T_2/4U$. Applying this inequality to (44) and (46) shows that $\overline{e}_{n1}^2 >> \overline{e}_{n2}^2$. When the input and output noise networks are combined, the two significant sources are the uncorrelated e_{n1} and i_{n2}. Thus, the low-noise amplifier noise correlation parameter is $q \approx 2$ in the unilateral case.

Generally, a low-noise amplifier network contains feedback elements, intended or not, and like the amplifier input circuit, these are limited to small losses. In this case, we turn to the invariant equation (6) to prove q-parameter behavior. Specifically, the right-hand side of this equation is a general invariant quantity that maintains constant value with applied lossless feedback. When feedback is negative, the denominator gain quantity is reduced, yet must remain near unity for a high-gain amplifier. Similarly, T_{emin} in the numerator is reduced, but must remain near the value of invariant $(MT_0)_{min}$ for a high-gain amplifier according to the expression for minimum noise measure given in Table I. When feedback is positive, both the denominator quantity and T_{emin} increase, but cannot exceed the upper bounds of unity and $(MT_0)_{min}$, respectively. Thus, feedback cannot change q in the numerator appreciably from the nominal unilateral value of 2. This completes the proof that noise correlation parameter $q \approx 2$ for any high-gain low-noise amplifier device.

The simplicity of the foregoing proof for $q \approx 2$ with feedback demonstrates the power of the network invariant equations. In the following Section IV, we will again use the invariant equations to find a lower bound on the q value of a high-gain cascade amplifier network. But first, we conclude Section III with a look at the effect of amplifier physical operating temperature on the q value and a summary of q-parameter behavior for a low-noise HEMT device.

D. q-Parameter of Low-Noise HEMT Device

Certainly, in the limit of physical temperature $T_1 = 0$, the basic amplifier of Fig. 10 has $q = 1$. However, the effect of the fundamental input/output decorrelation property is very strong, so that temperature much lower than typical cryogenic 10 to 20 K is necessary to lower q significantly from its room temperature value. This can be demonstrated using recent low-noise HEMT device models at ambient temperatures of 10 K and 300 K [15]. The device model is shown in Fig. 11. Element values were obtained by fitting measured amplifier noise temperature and gain data over 2–10 GHz to the device model integrated in a calibrated model of the amplifier. (Note that we have omitted parasitic elements at the gate and drain of the model for simplicity since q is invariant to them.) Fig. 12 shows computed q parameter for the low-noise device network of Fig. 11. The measured low-noise device confirms a small difference in q at 10 K and 300 K in accordance with network theory presented here.

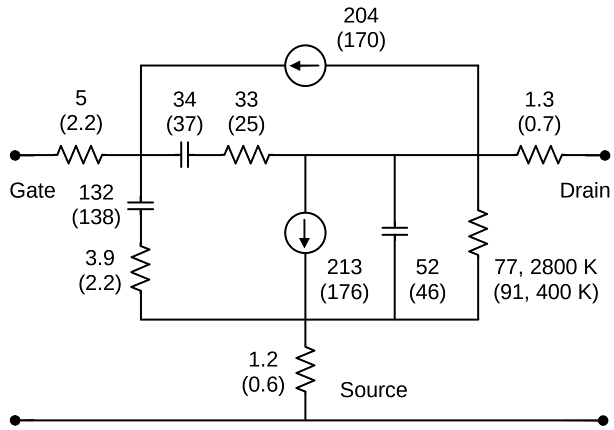

Fig. 11. Low-noise HEMT device model at ambient temperature 300 K (10 K) [15]. All resistor noise temperatures are ambient except drain–source resistor as shown. Units for resistance, capacitance, transconductance, and rms noise current density are, respectively: Ω, fF, mS, fA/(Hz)$^{1/2}$.

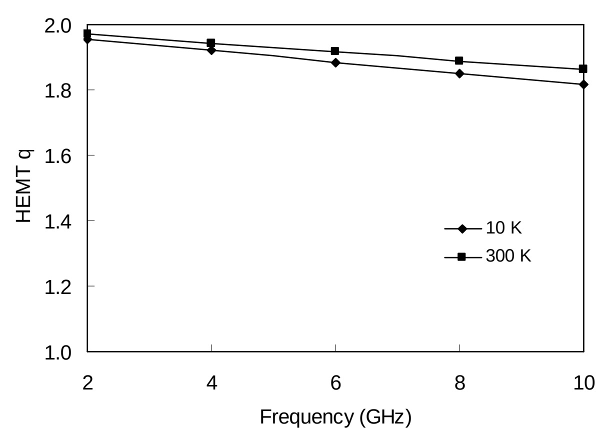

Fig. 12. Computed q parameter of the low-noise HEMT device network of Fig. 11 at ambient temperatures of 10 K and 300 K.

The network of Fig. 11, moreover, illustrates all the points of analyses and q-parameter behavior discussed in Sections III-B, C. The device includes several uncorrelated physical noise sources. One of these sources, the source-terminal resistor, has individual $q = \infty$, which means that it has $T_{emin} = 0$ at finite source impedance Z_s. All other device sources have individual $q = 1$, except the special case of gate-terminal resistor with a limiting value of $q = 2$. Since the source-terminal resistor is not the strongest noise source of the device, it is known immediately that the overall combined noise correlation parameter must be $q \leq 2$. The noise sources decorrelate to a q value just under 2 according to the input/output decorrelation property of low-noise devices. This we see in Fig. 12 at 2 GHz, where input circuit losses are low and unilateral gain is high. As frequency increases, parasitic input losses increase and unilateral gain decreases rapidly with a $1/f^2$ variation, yet the q value falls slowly with these frequency effects. Noise correlation parameter q is in a sense a measure of quality of a low-noise device. When the q value has fallen sufficiently from the ideal value of 2, we no longer have a low-noise device.

IV. Cascade Amplifier Characterization

In practice, a *high-gain cascade* of many individual amplifier stages is required to amplify very low-level signals to a usable power level for detection and processing. Since the power added noise temperature of a cascade of any number is identically equal to that of the individual amplifier, independent of gain or gain order, we have for the minimum noise temperature of a high-gain (infinite) cascade

$$T'_{emin} = (MT_0)'_{min} = (MT_0)_{min}\,. \tag{48}$$

Here and following, primed variables represent cascade quantities. Optimum source admittance for the cascade is that which gives minimum power added noise temperature of the individual amplifier

$$Y'_{opt} = Y'^{M}_{opt} = Y^{M}_{opt}\,. \tag{49}$$

The superscript "M" denotes optimum with respect to noise measure and power added noise temperature. Equations (48) and (49) are well-known results stated by Haus and Adler [5] in terms of noise measure.

System noise performance in a real environment is subject to source admittance variations that affect receiver noise temperature. It is of particular interest, therefore, to know the sensitivity of cascade noise temperature to source variation, and if the sensitivity can be minimized by appropriate design of the individual amplifier stages. In this section, we discuss cascade noise correlation parameter q' which defines cascade sensitivity and completes cascade amplifier characterization.

A. Cascade q-Parameter Lower Bound

Here, we find the lower bound of q-parameter for the optimum (i.e., minimum noise temperature) infinite cascade of identical amplifiers.

Invariant equation (6) written for the cascade is

$$(MT_0)'_{min}\, T'_{cmin} = \frac{(q'-1)T'^{2}_{emin}}{1-\dfrac{1}{G'_{mi}}}\,. \tag{50}$$

In view of (48), and since all forward gains of the cascade approach infinity, (50) reduces to

$$q' = 1 + \frac{T'_{cmin}}{(MT_0)_{min}}\,. \tag{51}$$

Now, lossless embedding of an arbitrary number of active two-port devices to form a new two-port (e.g., cascade network) reduces the number of ports and does not *necessarily* preserve eigenvalues of the original two-port device [9]. In other words, constraints on the cascade network generally result in a minimum cold load temperature greater, but never less than that of the constituent device. This is expressed by the inequality

$$T'_{cmin} \geq T_{cmin}\,. \tag{52}$$

Applying (52) to (51), we have

$$q' \geq 1 + \frac{T_{cmin}}{(MT_0)_{min}}\,. \tag{53}$$

Equation (53) gives the lower bound of cascade noise correlation parameter q' in terms of constituent device invariants. It shows that cascade noise sensitivity to source admittance cannot be reduced by any lossless device embedding. These results establish a fundamental performance constraint on high-gain detection and measurement systems.

B. Derivation of Cascade q-Parameter

Having found the lower bound, we now derive the *actual* cascade noise correlation parameter, again in terms of constituent amplifier invariants.

In addition to the cascade conditions (48) and (49), there is a required optimum cascade condition on amplifier output admittance

$$Y_{out} = Y^{M}_{opt}\,. \tag{54}$$

These relations are used to solve for q'. The equations needed are obtained from the correlation matrix relation given by Hillbrand and Russer [10, eq. (7)] for the cascade connection of two two-ports. In our case, a single amplifier (first two-port) is placed ahead of a high-gain cascade (second two-port). In the limit of infinite cascade stages, the correlation matrix of

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the resultant two-port is identically equal to that of the second two-port. Therefore, we can write

$$\mathbf{C}' = \mathbf{C} + \mathbf{T}\mathbf{C}'\mathbf{T}^{+} \tag{55}$$

where **T** and **C** are signal and noise matrices, respectively, of the single amplifier, given by (3) and (4). **C′** is the noise correlation matrix of the cascade, given by (4) with primed variables substituted.

After performing the matrix operations in (55) and equating the resulting matrix elements, we have a system of four equations with cascade unknowns q', T'_{emin}, and $Y'_{opt} = G'_{opt} + jB'_{opt}$. Substituting from (48), (49), and (54), we obtain for the cascade noise correlation parameter

$$q' = \frac{2 + \dfrac{(q-2)T_{emin}}{(MT_0)_{min}} - \dfrac{2k}{G_{ms}}}{1 - G_{pr}} \tag{56}$$

where G_{pr} is *reverse power gain* of an individual amplifier, defined as power delivered to the input termination divided by power entering the output port.

$$G_{pr} = \left| \frac{y_{12}}{y_{11} + Y^{M}_{opt}} \right|^2 \tag{57}$$

Note that (56) can be evaluated by individual amplifier quantities before the addition of an interstage network generally required by (54) since all amplifier quantities in (56) are invariant.

Inspection of (56) shows that $q' \approx 2$ under the following conditions typical of an individual low-noise amplifier stage:

$$q \approx 2 \tag{58}$$

$$T_{emin} \approx (MT_0)_{min} \tag{59}$$

$$\frac{2k}{G_{ms}} << 1 \tag{60}$$

$$G_{pr} << 1. \tag{61}$$

This is consistent with the general noise correlation parameter principles of Section III which apply equally well to an optimum low-noise cascade network.

C. *Simultaneous Noise and Power Match*

For best system performance, it is desirable to minimize the degradation of receiver noise temperature by source (e.g., antenna) impedance variation. Noise correlation parameter q' defines sensitivity of receiver noise temperature to source impedance variation. We now show that the q' lower bound (53) is realized by simultaneous noise and power match of the individual amplifier stages comprising an optimum cascade.

Simultaneous noise and power match means that amplifier minimum noise temperature and maximum available gain occur at the same source admittance. It follows from the definition of M_{min} (Table I) that minimum power added noise temperature is also realized at that same source admittance. Therefore, we have for such an amplifier

$$Y_{opt} = Y^{G}_{opt} = Y^{M}_{opt} \tag{62}$$

$$(MT_0)_{min} = \frac{T_{emin}}{1 - \dfrac{1}{G_{ma}}} \tag{63}$$

$$G_{aopt} = G_{ma}. \tag{64}$$

The superscript "G" denotes optimum with respect to available gain. According to the optimum cascade condition (54), amplifiers having simultaneous noise and power match operate in the cascade with conjugate match at the input and output ports. In this case, the reverse power gain G_{pr} is equal to reverse maximum available gain, and we have

$$G_{pr} = \frac{1/G_{ms}}{k + \sqrt{k^2 - 1}}. \tag{65}$$

Now, dividing (7) by $(MT_0)_{min}$ and rearranging, the q' lower bound can be expressed as

$$1 + \frac{T_{cmin}}{(MT_0)_{min}} = 2 - \frac{T_{emin}\Delta}{(MT_0)_{min}(1 - 1/G_{mi})}. \tag{66}$$

Using (63)–(65), we find the right-hand sides of (56) and (66) identically equal, thereby proving the realization of q' lower bound by simultaneous noise and power match.

We can summarize fundamental noise properties achieved by optimum cascade amplifier networks that include simultaneous noise and power match:

$$q'_{min} = 1 + \frac{T_{cmin}}{(MT_0)_{min}} \tag{67}$$

$$T'_{emin} = (MT_0)_{min} \tag{68}$$

$$N'_{min} = \frac{(MT_0)_{min} + T_{cmin}}{4T_0}. \tag{69}$$

Noise temperature equation (23) in this case becomes

$$T_e' = (MT_0)_{\min} + \left[(MT_0)_{\min} + T_{c\min}\right]\frac{(S-1)^2}{4S}. \qquad (70)$$

Thus, device invariants $(MT_0)_{\min}$ *and* $T_{c\min}$ *define minimum sensitivity of noise temperature to source impedance variation as well as minimum noise temperature of a high-gain cascade.*

As we know, the minimum noise temperature given in (68) applies not only to a high-gain cascade, but also to a single device at high gain by positive feedback. Likewise, in the limits $G_{mi} \to \infty$ and $T_{e\min} \to (MT_0)_{\min}$ for a high-gain single device, (6) shows that q is the minimum given in (67). Noise equation (70) shows explicitly how both general invariants of a device define best possible noise temperature and source impedance sensitivity performance of a high-gain system.

V. Conclusion

In this paper, we have extended the work of Haus and Adler [5], [9] to provide a complete and unified theory of linear noisy two-ports. Two general noise temperature (power) invariants of noisy two-ports have been identified as minimum power added noise temperature and minimum cold load temperature. The first invariant defines minimum possible noise temperature of a high-gain cascade of individual two-ports—the well-known noise measure relation. In addition, we found that both general invariants combine to define minimum possible sensitivity of cascade noise temperature to source impedance. No embedding applied to the individual two-port or to the cascade can lower this minimum sensitivity. It was shown that a high-gain cascade amplifier network having simultaneous minimum noise temperature and minimum noise temperature sensitivity is realized by simultaneous noise and power match of the individual amplifier stages. We introduced a two-port noise correlation parameter q that determines amplifier noise sensitivity to source impedance, and showed that $q \approx 2$ for any low-noise device or cascade. Thus, noise temperature sensitivity is approximately fixed and cannot be improved by amplifier or semiconductor device design.

Associated with the general noise invariants of two-port networks are two invariant equations that relate network noise parameter invariants to gain and stability signal invariants. These are the hallmark of a "unified" network noise theory. The invariant equations and the *a priori* knowledge of noise correlation parameter $q \approx 2$ can provide useful information to application of two-port characterization in many areas of interest. For example, in the contemporary field of wideband active receiving arrays, the active (source) impedance presented to a low-noise amplifier can vary significantly with frequency and scan angle. This is particularly important for systems in which receiver noise dominates, notably in radio astronomy, and here, the network condition $q \approx 2$ can be applied to system design and optimization methods. (For example, $q \approx 2$ *must* be applied in [16], [17] to obtain valid results.) For noise measurement systems and noise parameter extraction, $q \approx 2$ in effect eliminates one variable. HEMT q values presented in the paper provide an estimate for FET devices. Bipolar transistor devices possess an inherent *RC* network at the input that reduces q slightly, compared to FETs, as explained by the network theory.

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TABLE I
KNOWN INVARIANTS OF LINEAR NOISY TWO-PORTS

Symbol	Name	Expression	Type
U	unilateral gain	$\frac{\lvert y_{21}-y_{12}\rvert^2}{4(\mathrm{Re}[y_{11}]\mathrm{Re}[y_{22}]-\mathrm{Re}[y_{12}]\mathrm{Re}[y_{21}])}$	general
M_{min}	minimum noise measure	$\left(\frac{F-1}{1-1/G_a}\right)_{\mathrm{min}}$	general
$T_{e\mathrm{min}}$	minimum noise temperature	noise parameter	special
G_{ma}	maximum available gain	$\frac{G_{ms}}{k+\sqrt{k^2-1}}$	special
$G_{a\mathrm{opt}}$	associated gain	$\frac{\lvert y_{21}\rvert^2 G_{\mathrm{opt}}}{\lvert y_{11}+Y_{\mathrm{opt}}\rvert^2\mathrm{Re}[y_{22}]-\mathrm{Re}[y_{12}y_{21}(y_{11}^*+Y_{\mathrm{opt}}^*)]}$	special
G_{ms}	maximum stable gain	$\left\lvert\frac{y_{21}}{y_{12}}\right\rvert$	special
k	stability factor	$\frac{2\mathrm{Re}[y_{11}]\mathrm{Re}[y_{22}]-\mathrm{Re}[y_{12}y_{21}]}{\lvert y_{12}y_{21}\rvert}$	special

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TABLE II
NEW INVARIANTS OF LINEAR NOISY TWO-PORTS

Symbol	Name	Expression	Type
$(MT_0)_{\min}$	minimum power added noise temperature	$\dfrac{T_{e\min}\left[\sqrt{4(q-1)(1-1/G_{mi})+\Delta^2}+\Delta\right]}{2(1-1/G_{mi})}$	general
$T_{c\min}$	minimum cold load temperature	$\dfrac{T_{e\min}\left[\sqrt{4(q-1)(1-1/G_{mi})+\Delta^2}-\Delta\right]}{2(1-1/G_{mi})}$	general
q	noise correlation parameter	$\dfrac{4R_n G_{\text{opt}} T_0}{T_{e\min}}$	special
q'	cascade noise correlation parameter	$\dfrac{2+\dfrac{(q-2)T_{e\min}}{(MT_0)_{\min}}-\dfrac{2k}{G_{ms}}}{1-G_{pr}}$	special
G_{mi}	maximum invariant gain	$\dfrac{G_{ms}}{2k-\dfrac{1}{G_{ms}}}$	special
G_{pr}	reverse power gain	$\left\lvert\dfrac{y_{12}}{y_{11}+Y_{\text{opt}}^M}\right\rvert^2$	special
$q'_{\min}$	minimum cascade noise correlation parameter	$1+\dfrac{T_{c\min}}{(MT_0)_{\min}}$	general
$N'_{\min}$	minimum cascade *N*-parameter	$\dfrac{(MT_0)_{\min}+T_{c\min}}{4T_0}$	general